A Mamdani Type Fuzzy Logic Controller

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1. Introduction

The database of a rule-based system may contain imprecisions which appear in the description of the rules given by the expert. Because such an inference can not be made by the methods which use classical two valued logic or many valued logic, Zadeh in (Zadeh, 1975) and Mamdani in (Mamdani, 1977) suggested an inference rule called "compositional rule of inference". Using this inference rule, several methods for fuzzy reasoning were proposed. Zadeh (Zadeh, 1979) extends the traditional Modus Ponens rule in order to work with fuzzy sets, obtaining the Generalized Modus Ponens (GMP) rule.

An important part of fuzzy reasoning is represented by Fuzzy Logic Control (FLC), derived from control theory based on mathematical models of the open-loop process to be controlled. Fuzzy Logic Control has been successfully applied to a wide variety of practical problems: control of warm water, robot, heat exchange, traffic junction, cement kiln, automobile speed, automotive engineering, model car parking and turning, power system and nuclear reactor, on-line shopping, washing machines, etc.

It points out that fuzzy control has been effectively used in the context of complex ill-defined processes, especially those that can be controlled by a skilled human operator without the knowledge of their underlying dynamics. In this sense, neural and adaptive fuzzy systems has been compared to classical control methods by B. Kosko in (Kosko, 1992). There, it is remarked that they are model-free estimators, i.e., they estimate a function without requiring a mathematical description of how the output functionally depends on the input; they learn from samples. However, some people criticized fuzzy control because the very fundamental question "Why does a fuzzy rule-based system have such good performance for a wide variety of practical problems?" remained unanswered. A first approach to answer this fundamental question in a quantitative way was presented by Wang in (Wang, 1992) where he proved that a particular class of FLC systems are universal approximators: they are capable of approximating any real continuous function on a compact set to arbitrary accuracy. This class is characterized by:

1) Gaussian membership functions,

2) Product fuzzy conjunction,

3) Product fuzzy implication,

4) Center of area defuzzification.

Other approaches are due to Buckley (Buckley, 1992; 1993). He has proved that a modification of Sugeno type fuzzy controllers gives universal approximators. Although both results are very important, many real fuzzy logic controllers do not belong to these classes, because other membership functions are used, other inference mechanisms are applied or other type of rules are used. The question "What other types of fuzzy logic controllers are universal approximators?" still remained unanswered. This problem were solved by Castro in (Castro, 1995) where he proved that a large number of classes of FLC systems are also universal approximators.

The most popular FLC systems are: Mamdani, Tsukamoto, Sugeno and Larsen which work with crisp data as inputs. An extension of the Mamdani model in order to work with interval inputs is presented in (Liu et al., 2005) , where the fuzzy sets are represented by triangular fuzzy numbers and the firing level of the conclusion is computed as the product of firing levels from the antecedent. Other extensions and applications of the standard FLC systems were proposed in (Iancu, 2009a;b; Iancu, Colhon & Dupac, 2010; Iancu, Constantinescu & Colhon, 2010; Iancu & Popirlan, 2010).

The necessity to extend the fuzzy controllers to work with intervals or linguistic values as inputs is given by many applications where precise values of the input data no interest or are difficult to estimate. For example, in shopping applications the buyer is interested, rather, in a product that is priced within certain limits or does not exceed a given value (Liu et al., 2005). In other cases, the input values are much easier to express in fuzzy manner, for example, in the problem of controlling the washing time using fuzzy logic control the degree of dirt for the object to be washed is easily expressed by a linguistic value (Agarwal, 2007). These examples will be used to show the working of the model proposed in order to expand the Mamdani fuzzy logic controller. In this paper a FLC system with the following characteristics is presented:

- the linguistic terms (or values) are represented by trapezoidal fuzzy numbers
- various implication operators are used to represent the rules
- the crisp control action of a rule is computed using Middle-of-Maxima method
- the overall crisp control action of an implication is computed by discrete Center-of-Gravity
- the overall crisp control action of the system is computed using an OWA (Ordered Weighted Averaging) operator.

2. Preliminaries

Let U be a collection of objects denoted generically by $\{u\}$, which could be discrete or continuous. U is called the universe of discourse and u represents the generic element of U.

Definition 1. *A fuzzy set F in the universe of discourse U is characterized by its membership function* $\mu_F : U \rightarrow [0,1]$. *The fuzzy set may be represented as a set of ordered pairs of a generic element u and its grade of membership function:* $F = \{(u, \mu_F(u))/u \in U\}$.

Definition 2. *A fuzzy number F in a continuous universe U, e. g., a real line, is a fuzzy set F in U which is normal and convex, i. e.,*

$$\max_{u \in U} \mu_F(u) = 1 \qquad (normal)$$

$$\mu_F(\lambda u_1 + (1 - \lambda)u_2) \geq min\{\mu_F(u_1), \mu_F(u_2)\}, \quad u_1, u_2 \in U, \lambda \in [0, 1] \qquad (convex)$$

Because the majority of practical applications work with trapezoidal or triangular distributions and these representations are still a subject of various recent papers ((Grzegorzewski & Mrowka, 2007), (Nasseri, 2008), for instance) we will work with membership functions represented by trapezoidal fuzzy numbers. Such a number $N = (\underline{m}, \overline{m}, \alpha, \beta)$ is defined as

$$\mu_N(x) = \begin{cases} 0 & for \ \ x < \underline{m} - \alpha \\ \dfrac{x - \underline{m} + \alpha}{\alpha} & for \ \ x \in [\underline{m} - \alpha, \underline{m}] \\ 1 & for \ \ x \in [\underline{m}, \overline{m}] \\ \dfrac{\overline{m} + \beta - x}{\beta} & for \ \ x \in [\overline{m}, \overline{m} + \beta] \\ 0 & for \ \ x > \overline{m} + \beta \end{cases}$$

Will be used fuzzy sets to represent linguistic variables. A linguistic variable can be regarded either as a variable whose value is a fuzzy number or as a variable whose values are defined in linguistic terms.

Definition 3. *A linguistic variable V is characterized by: its name x, an universe U, a term set $T(x)$, a syntactic rule G for generating names of values of x, and a set of semantic rule M for associating with each value its meaning.*

For example, if *speed* of a car is interpreted as a linguistic variable, then its term set could be $T(x) = \{slow, moderate, fast, very \ slow, more \ or \ less \ fast\}$ where each term is characterized by a fuzzy set in an universe of discourse $U = [0, 100]$. We might interpret: *slow* as "a speed below about 40 mph", *moderate* as "speed close to 55 mph", *fast* as "a speed about 70 mph".

Definition 4. *A function $T : [0, 1]^2 \rightarrow [0, 1]$ is a t-norm iff it is commutative, associative, non-decreasing and $T(x, 1) = x \ \forall x \in [0, 1]$.*

The most important t-norms are:

- Minimum: $T_m(x, y) = min\{x, y\}$

- Lukasiewicz: $T_L(x, y) = max\{0, x + y - 1\}$

- Probabilistic (or Product): $T_P(x, y) = xy$

- Weak: $T_W(x, y) = \begin{cases} min\{x, y\} & if \ \ max\{x, y\} = 1 \\ 0 & otherwise. \end{cases}$

Definition 5. *A function $S : [0, 1]^2 \rightarrow [0, 1]$ is a t-conorm iff it is commutative, associative, non-decreasing and $S(x, 0) = x \ \forall x \in [0, 1]$.*

The basic t-conorms are

- Maximum: $S_m(x, y) = max\{x, y\}$

- Lukasiewicz: $S_L(x, y) = min\{1, x + y\}$

- Probabilistic (or Product): $S_P(x, y) = x + y - xy$

- Strong: $S_S(x,y) = \begin{cases} max\{x,y\} \; if & min\{x,y\} = 1 \\ 1 & otherwise. \end{cases}$

The t-norms are used to compute the firing levels of the rules or as aggregation operators and the t-conorms are used as aggregation operators. The rules are represented by fuzzy implications. Let X and Y be two variables whose domains are U and V, respectively. A causal link from X to Y is represented as a conditional possibility distribution ((Zadeh, 1979), (Zadeh, 1978)) $\pi_{Y/X}$ which restricts the possible values of Y for a given value of X. For the rule

$$IF \; X \; is \; A \; THEN \; Y \; is \; B$$

we have

$$\forall u \in U, \; \forall v \in V, \; \pi_{Y/X}(v,u) = \mu_A(u) \rightarrow \mu_B(v)$$

where $\rightarrow$ is an implication operator and μ_A and μ_B are the membership functions of the fuzzy sets A and B, respectively.

Definition 6. *An implication is a function* $I : [0,1]^2 \rightarrow [0,1]$ *satisfying the following conditions for all* $x,y,z \in [0,1]$:

I1: *If* $x \leq z$ *then* $I(x,y) \geq I(z,y)$

I2: *If* $y \leq z$ *then* $I(x,y) \leq I(x,z)$

I3: $I(0,y) = 1$ *(falsity implies anything)*

I4: $I(x,1) = 1$ *(anything implies tautology)*

I5: $I(1,0) = 0$ *(Booleanity).*

The following properties could be important in some applications:

I6: $I(1,x) = x$ (tautology cannot justify anything)

I7: $I(x,I(y,z)) = I(y,I(x,z))$ (exchange principle)

I8: $x \leq y$ if and only if $I(x,y) = 1$ (implication defines ordering)

I9: $I(x,0) = N(x)$ is a strong negation

I10: $I(x,y) \geq y$

I11: $I(x,x) = 1$ (identity principle)

I12: $I(x,y) = I(N(y),N(x))$, where N is a strong negation

I13: I is a continuous function.

The most important implications are:
Willmott: $I_W(x,y) = \max\{1-x, \min\{x,y\}\}$

Mamdani: $I_M(x,y) = \min\{x,y\}$

Rescher-Gaines: $I_{RG}(x,y) = \begin{cases} 1 \; if & x \leq y \\ 0 \; otherwise \end{cases}$

Kleene-Dienes: $I_{KD}(x,y) = \max\{1-x, y\}$

Brouwer-Gödel: $\quad I_{BG}(x,y) = \begin{cases} 1 \ if \quad x \le y \\ y \ otherwise \end{cases}$

Goguen: $\quad I_G(x,y) = \begin{cases} 1 \ if \quad x \le y \\ \frac{y}{x} \ otherwise \end{cases}$

Lukasiewicz: $\quad I_L(x,y) = \min\{1-x+y,1\}$

Fodor: $\quad I_F(x,y) = \begin{cases} 1 \qquad if \quad x \le y \\ max\{1-x,y\} \ otherwise \end{cases}$

Reichenbach: $\quad I_R(x,y) = 1 - x + xy.$

Definition 7. *An n-ary fuzzy relation is a fuzzy set in $U_1 \times U_2 \times \cdots \times U_n$ expressed as*

$$R_{U_1 \times \cdots \times U_n} = \{((u_1, \cdots, u_n), \mu_R(u_1, \cdots, u_n))/(u_1, \cdots, u_n) \in U_1 \times \cdots \times U_n\}.$$

Definition 8. *If R and S are fuzzy relations in $U \times V$ and $V \times W$, respectively, then the sup-star composition of R and S is a fuzzy relation denoted by $R \circ S$ and defined by*

$$R \circ S = \{[(u,w), \sup_v (\mu_R(u,v) * \mu_S(v,w))]/u \in U, v \in V, w \in W\}$$

where $*$ can be any operator in the class of t-norms.

Fuzzy implication inference is based on the compositional rule of inference for approximate reasoning suggested by Zadeh in (Zadeh, 1973).

Definition 9. *If R is a fuzzy relation on $U \times V$ and x is a fuzzy set in U then the "sup-star compositional rule of inference" asserts that the fuzzy set y in V induced by x is given by (Zadeh, 1971)*

$$y = x \circ R$$

where $x \circ R$ is the sup-star composition of x and R.

If the star represents the minimum operator then this definition reduces to Zadeh's compositional rule of inference (Zadeh, 1973).

The process of information aggregation appears in many applications related to the development of intelligent systems: fuzzy logic controllers, neural networks, vision systems, expert systems, multi-criteria decision aids. In (Yager, 1988) Yager introduced an aggregation technique based on OWA operators.

Definition 10. *An OWA operator of dimension n is a mapping $F : R^n \to R$ that has an associated n vector $w = (w_1, w_2, ..., w_n)^t$ such as*

$$w_i \in [0,1], 1 \le i \le n, \sum_{i=1}^n w_i = 1.$$

The aggregation operator of the values $\{a_1, a_2, ..., a_n\}$ is

$$F(a_1, a_2, ..., a_n) = \sum_{j=1}^n w_j b_j$$

where b_j is the j-th largest element from $\{a_1, a_2, ..., a_n\}$.

It is sufficiently to work with rules with a single conclusion because a rule with multiple consequent can be treated as a set of such rules.

3. Standard fuzzy logic controllers

3.1 Structure of a fuzzy logic controller

The seminal work by L.A. Zadeh (Zadeh, 1973) on fuzzy algorithms introduced the idea of formulating the control algorithm by logical rules. An FLC consists of a set of rules of the form

$$IF \; (a \; set \; of \; conditions \; are \; satisfied) \; THEN \; (a \; set \; of \; consequences \; can \; be \; inferred).$$

Since the antecedents and the consequents of these IF-THEN rules are associated with fuzzy concepts (linguistic terms), they are often called *fuzzy conditional statements*. In FLC terminology, a *fuzzy control rule* is a fuzzy conditional statement in which the antecedent is a condition in its application domain and the consequent is a control action for the system under control. The inputs of fuzzy rule-based systems should be given by fuzzy sets, and therefore, we have to fuzzify the crisp inputs. Furthermore, the output of a fuzzy system is always a fuzzy set, and therefore to get crisp value we have to defuzzify it. Fuzzy logic control systems usually consist of four major parts: Fuzzification interface, Fuzzy rule base, Fuzzy inference engine and Defuzzification interface, as is presented in the Figure 1.
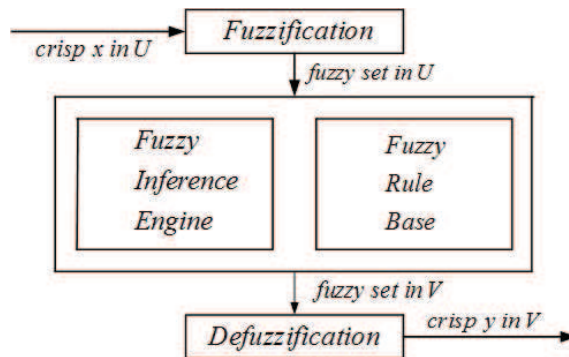

Fig. 1. Fuzzy Logic Controller

The four components of a FLC are explained in the following (Lee, 1990).

The *fuzzification interface* involves the functions:

a) measures the values of inputs variables,

b) performs a scale mappings that transfers the range of values of inputs variables into corresponding universes of discourse,

c) performs the function of fuzzyfication that converts input data into suitable linguistic values which may be viewed as label of fuzzy sets.

The *rule base* comprises a knowledge of the application domain and the attendant control goals. It consists of a "data base" and a "linguistic (fuzzy) control rule base":

a) the data base provides necessary definitions which are used to define linguistic control rules and fuzzy data manipulation in a FLC

b) the rule base characterizes the control goals and the control policy of the domain experts by means of a set of linguistic control rules.

The *fuzzy inference engine* is the kernel of a FLC; it has the capability of simulating human decision-making based of fuzzy concepts and of inferring fuzzy control actions employing fuzzy implication and the rules of inference in fuzzy logic.

The *defuzzification interface* performs the following functions:

a) a scale mapping, which converts the range of values of output variables into corresponding universes of discourse

b) defuzzification, which yields a non fuzzy control action from an inferred fuzzy control action.

A fuzzification operator has the effect of transforming crisp data into fuzzy sets. In most of the cases fuzzy singletons are used as fuzzifiers (according to Figure 2).
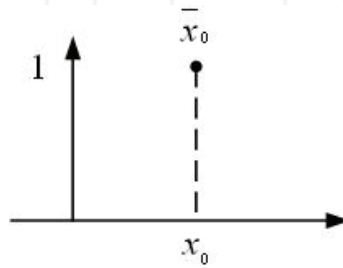

Fig. 2. Fuzzy singleton as fuzzifier

In other words,

$$fuzzifier(x_0) = \overline{x}_0,$$

$$\mu_{\overline{x}_0}(x) = \begin{cases} 1 & for \ \ x = x_0 \\ 0 & for \ \ x \neq x_0 \end{cases}$$

where x_0 is a crisp input value from a process.

The procedure used by Fuzzy Inference Engine in order to obtain a fuzzy output consists of the following steps:

1. find the firing level of each rule,

2. find the output of each rule,

3. aggregate the individual rules outputs in order to obtain the overall system output.

The fuzzy control action C inferred from the fuzzy control system is transformed into a crisp control action:

$$z_0 = defuzzifier(C)$$

where *defuzzifier* is a defuzzification operator. The most used defuzzification operators, for a discrete fuzzy set C having the universe of discourse V, are:

• Center-of-Gravity:

$$z_0 = \frac{\sum\limits_{j=1}^{N} z_j \mu_C(z_j)}{\sum\limits_{j=1}^{N} \mu_C(z_j)}$$

• Middle-of-Maxima: the defuzzified value is defined as mean of all values of the universe of discourse, having maximal membership grades

$$z_0 = \frac{1}{N1} \sum_{j=1}^{N1} z_j, \quad N1 \leq N$$

• Max-Criterion: this method chooses an arbitrary value, from the set of maximizing elements of C, i. e.

$$z_0 \in \{z / \mu_C(z) = \max_{v \in V} \mu_C(v)\},$$

where $Z = \{z_1, ..., z_N\}$ is a set of elements from the universe V.

Because several linguistic variables are involved in the antecedents and the conclusions of a rule, the fuzzy system is of the type multi–input–multi–output. Further, the working with a FLC for the case of a two-input-single-output system is explained. Such a system consists of a set of rules

$$\text{R}_1 : \ IF \ x \ is \ A_1 \ AND \ y \ is \ B_1 \ THEN \ z \ is \ C_1$$

$$\text{R}_2 : \ IF \ x \ is \ A_2 \ AND \ y \ is \ B_2 \ THEN \ z \ is \ C_2$$

$$...$$

$$\text{R}_n : \ IF \ x \ is \ A_n \ AND \ y \ is \ B_n \ THEN \ z \ is \ C_n$$

and a set of inputs

$$\texttt{fact} : \ x \ is \ x_0 \ AND \ y \ is \ y_0$$

where x and y are the process state variables, z is the control variable, A_i, B_i and C_i are linguistic values of the linguistic variables x, y and z in the universes of discourse U, V and W, respectively. Our task is to find a crisp control action z_0 from the fuzzy rule base and from the actual crisp inputs x_0 and y_0. A fuzzy control rule

$$\text{R}_i : \ IF \ x \ is \ A_i \ AND \ y \ is \ B_i \ THEN \ z \ is \ C_i$$

is implemented by a fuzzy implication I_i and is defined as

$$\mu_{I_i}(u, v, w) = [\mu_{A_i}(u) \ AND \ \mu_{B_i}(v)] \rightarrow \mu_{C_i}(w) = T(\mu_{A_i}(u), \mu_{B_i}(v)) \rightarrow \mu_{C_i}(w)$$

where T is a t-norm used to model the logical connective AND. To infer the consequence "z is C" from the set of rules and the facts, usually the compositional rule of inference is applied; it gives

$$consequence = Agg\{fact \circ \text{R}_1, ..., fact \circ \text{R}_n\}.$$

That is

$$\mu_C = Agg\{T(\mu_{\bar{x}_0}, \mu_{\bar{y}_0}) \circ R_1, ..., T(\mu_{\bar{x}_0}, \mu_{\bar{y}_0}) \circ R_n\}.$$

Taking into account that $\mu_{\bar{x}_0}(u) = 0$ for $u \neq x_0$ and $\mu_{\bar{y}_0}(v) = 0$ for $v \neq y_0$, the membership function of C is given by

$$\mu_C(w) = Agg\{T(\mu_{A_1}(x_0), \mu_{B_1}(y_0)) \rightarrow \mu_{C_1}(w), ..., T(\mu_{A_n}(x_0), \mu_{B_n}(y_0)) \rightarrow \mu_{C_n}(w)\}$$

for all $w \in W$.

The procedure used for obtaining the fuzzy output from a FLC system is

- the firing level of the i-th rule is determined by

$$T(\mu_{A_i}(x_0), \mu_{B_i}(y_0))$$

- the output C_i' of the i-th rule is given by

$$\mu_{C_i'}(w) = T(\mu_{A_i}(x_0), \mu_{B_i}(y_0)) \rightarrow \mu_{C_i}(w)$$

- the overall system output, C, is obtained from the individual rule outputs, by aggregation operation:

$$\mu_C(w) = Agg\{\mu_{C_1'}(w), ..., \mu_{C_n'}(w)\}$$

for all $w \in W$.

3.2 Mamdani fuzzy logic controller

The most commonly used fuzzy inference technique is the so-called Mamdani method (Mamdani & Assilian, 1975) which was proposed, by Mamdani and Assilian, as the very first attempt to control a steam engine and boiler combination by synthesizing a set of linguistic control rules obtained from experienced human operators. Their work was inspired by an equally influential publication by Zadeh (Zadeh, 1973). Interest in fuzzy control has continued ever since, and the literature on the subject has grown rapidly. A survey of the field with fairly extensive references may be found in (Lee, 1990) or, more recently, in (Sala et al., 2005). In Mamdani's model the fuzzy implication is modeled by Mamdani's minimum operator, the conjunction operator is *min*, the t-norm from compositional rule is *min* and for the aggregation of the rules the *max* operator is used. In order to explain the working with this model of FLC will be considered the example from (Rakic, 2010) where a simple two-input one-output problem that includes three rules is examined:

$$Rule1: \quad IF\ x\ is\ A_3\ OR\ y\ is\ B_1\ THEN\ z\ is\ C_1$$

$$Rule2: \quad IF\ x\ is\ A_2\ AND\ y\ is\ B_2\ THEN\ z\ is\ C_2$$

$$Rule3: \quad IF\ x\ is\ A_1\ THEN\ z\ is\ C_3.$$

Step 1: Fuzzification

The first step is to take the crisp inputs, x_0 and y_0, and determine the degree to which these inputs belong to each of the appropriate fuzzy sets. According to Fig 3(a) one obtains

$$\mu_{A_1}(x_0) = 0.5, \mu_{A_2}(x_0) = 0.2, \mu_{B_1}(y_0) = 0.1, \mu_{B_2}(y_0) = 0.7$$

Step 2: Rules evaluation

The fuzzified inputs are applied to the antecedents of the fuzzy rules. If a given fuzzy rule has multiple antecedents, the fuzzy operator (AND or OR) is used to obtain a single number that represents the result of the antecedent evaluation. To evaluate the disjunction of the rule antecedents, one uses the OR fuzzy operation. Typically, the classical fuzzy operation union is used :

$$\mu_{A \cup B}(x) = max\{\mu_A(x), \mu_B(x)\}.$$

Similarly, in order to evaluate the conjunction of the rule antecedents, the AND fuzzy operation intersection is applied:

$$\mu_{A \cap B}(x) = min\{\mu_A(x), \mu_B(x)\}.$$

The result is given in the Figure 3(b).

Now the result of the antecedent evaluation can be applied to the membership function of the consequent. The most common method is to cut the consequent membership function at the level of the antecedent truth; this method is called *clipping*. Because top of the membership function is sliced, the clipped fuzzy set loses some information. However, clipping is preferred because it involves less complex and generates an aggregated output surface that is easier to defuzzify. Another method, named *scaling*, offers a better approach for preserving the original shape of the fuzzy set: the original membership function of the rule consequent is adjusted by multiplying all its membership degrees by the truth value of the rule antecedent (see Fig. 3(c)).

Step 3: Aggregation of the rule outputs

The membership functions of all rule consequents previously clipped or scaled are combined into a single fuzzy set (see Fig. 4(a)).

Step 4: Defuzzification

The most popular defuzzification method is the centroid technique. It finds a point representing the center of gravity (COG) of the aggregated fuzzy set A, on the interval $[a, b]$. A reasonable estimate can be obtained by calculating it over a sample of points. According to Fig. 4(b), in our case results

$$COG = \frac{(0 + 10 + 20) \times 0.1 + (30 + 40 + 50 + 60) \times 0.2 + (70 + 80 + 90 + 100) \times 0.5}{0.1 + 0.1 + 0.1 + 0.2 + 0.2 + 0.2 + 0.2 + 0.5 + 0.5 + 0.5 + 0.5} = 67.4$$

3.3 Universal approximators

Using the Stone-Weierstrass theorem, Wang in (Wang, 1992) showed that fuzzy logic control systems of the form

$$R_i : \ IF \ x \ is \ A_i \ AND \ y \ is \ B_i \ THEN \ z \ is \ C_i, \quad i = 1, ..., n$$

with

(a) Fuzzification

Rule 1:   IF   x   is A_3 (0.0)   OR   Y   is B_1 (0.1)   THEN   z   is C_1 (0.1)

Rule 2:   IF   x   is A_2 (0.2)   AND   y   is B_2 (0.7)   THEN   z   is C_2 (0.2)

Rule 3:   IF   x   is A_1 (0.5)   THEN   z   is C_3 (0.5)

(b) Rules evaluation

(c) Clipping and scaling
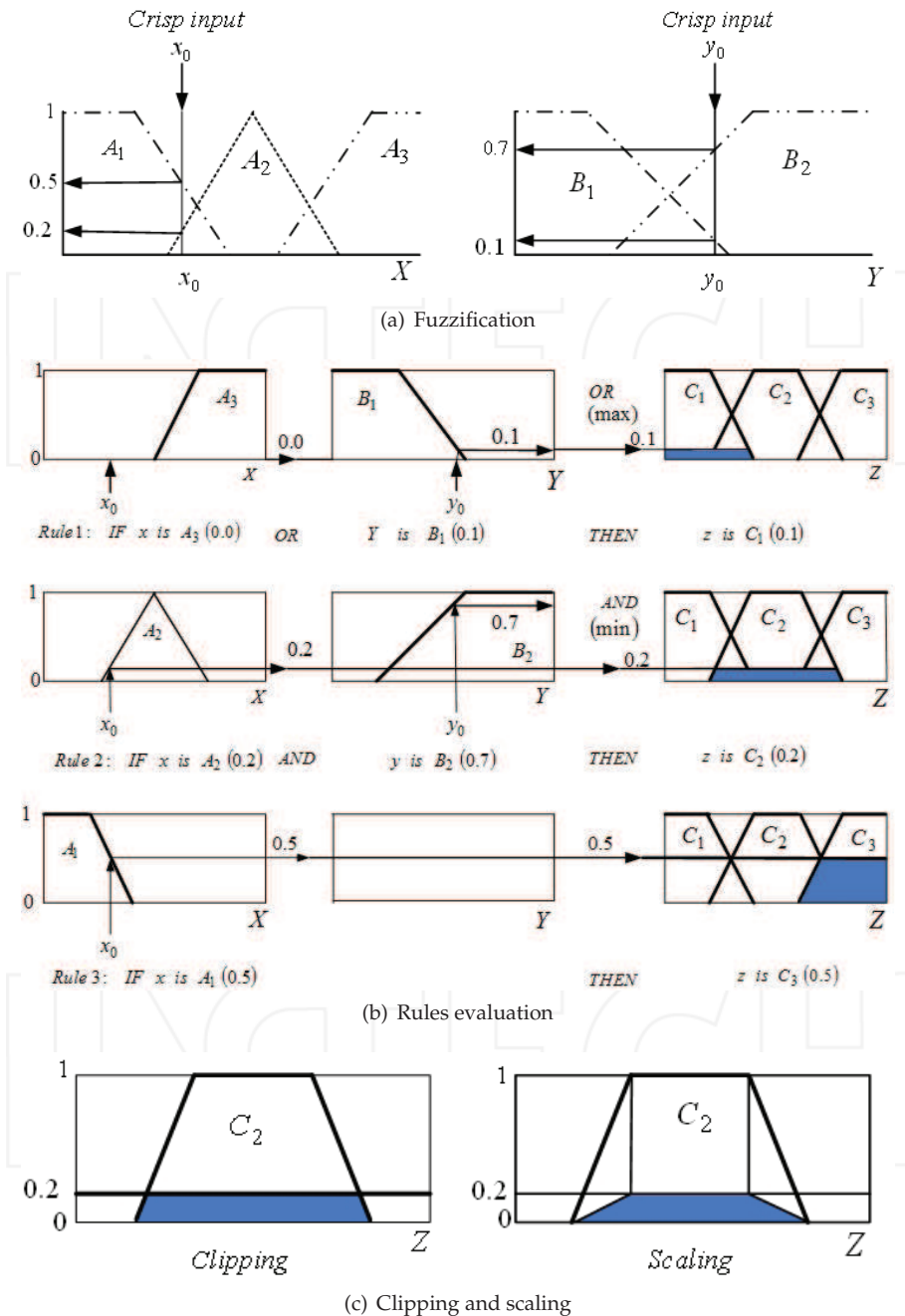
Fig. 3. Mamdani fuzzy logic controller
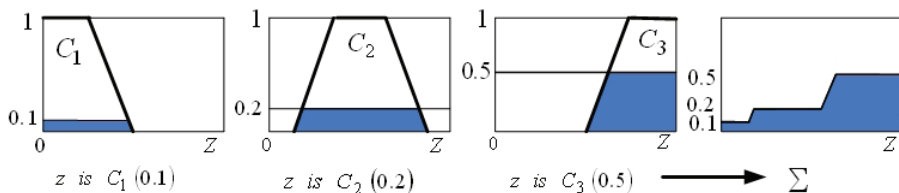
(a) Aggregation of the rule outputs
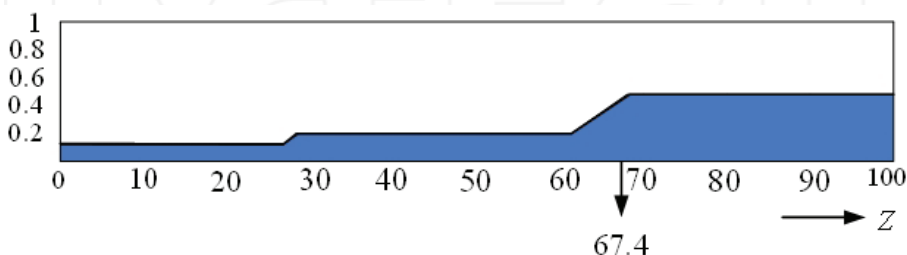
(b) Defuzzification

Fig. 4. Mamdani fuzzy logic controller

- Gaussian membership functions

$$\mu_A(x) = \exp[-\frac{1}{2}(\frac{x - x_0}{\sigma})^2]$$

  where x_0 is the position of the peak relative to the universe and σ is the standard deviation

- Singleton fuzzifier

$$fuzzifier(x) = \bar{x}$$

- Fuzzy product conjunction

$$\mu_{A_i}(u) \; AND \; \mu_{B_i}(v) = \mu_{A_i}(u)\mu_{B_i}(v)$$

- Larsen (fuzzy product) implication

$$[\mu_{A_i}(u) \; AND \; \mu_{B_i}(v)] \rightarrow \mu_{C_i}(w) = \mu_{A_i}(u)\mu_{B_i}(v)\mu_{C_i}(w)$$

- Centroid deffuzification method

$$z = \frac{\sum_{i=1}^{n} c_i\mu_{A_i}(x)\mu_{B_i}(y)}{\sum_{i=1}^{n} \mu_{A_i}(x)\mu_{B_i}(y)}$$

  where c_i is the center of C_i, are universal approximators, i.e. they can approximate any continuous function on a compact set to an arbitrary accuracy.

More generally, Wang proved the following theorem

Theorem 1. *For a given real-valued continuous function g on the compact set U and arbitrary $\epsilon > 0$, there exists a fuzzy logic control system with output function f such that*

$$\sup_{x \in U} \|g(x) - f(x)\| \leq \epsilon.$$

Castro in (Castro, 1995) showed that Mamdani fuzzy logic controllers

$$R_i : \ IF \ x \ is \ A_i \ AND \ y \ is \ B_i \ THEN \ z \ is \ C_i, i = 1, ..., n$$

with

- Symmetric triangular membership functions

$$\mu_A(x) = \begin{cases} 1 - \dfrac{x - a}{\alpha} \ if \ \ |x - a| \leq \alpha \\ \quad 0 \qquad \qquad otherwise \end{cases}$$

- Singleton fuzzifier

$$fuzzifier(x_0) = \bar{x}_0$$

- Minimum norm fuzzy conjunction

$$\mu_{A_i}(u) \ AND \ \mu_{B_i}(v) = min\{\mu_{A_i}(u), \mu_{B_i}(v)\}$$

- Minimum-norm fuzzy implication

$$[\mu_{A_i}(u) \ AND \ \mu_{B_i}(v)] \to \mu_{C_i}(w) = min\{\mu_{A_i}(u), \mu_{B_i}(v), \mu_{C_i}(w)\}$$

- Maximum t-conorm rule aggregation

$$Agg\{R_1, R_2, ..., R_n\} = max\{R_1, R_2, ..., R_n\}$$

- Centroid defuzzification method

$$z = \frac{\sum_{i=1}^{n} c_i min\{\mu_{A_i}(x), \mu_{B_i}(y)\}}{\sum_{i=1}^{n} min\{\mu_{A_i}(x), \mu_{B_i}(y)\}}$$

where c_i is the center of C_i, are universal approximators.

More generally, Castro (Castro, 1995) studied the following problem:

Given a type of FLC, (i.e. a fuzzification method, a fuzzy inference method, a defuzzification method, and a class of fuzzy rules, are fixed), an arbitrary continuous real valued function f on a compact $U \subset R^n$, and a certain $\epsilon > 0$, is it possible to find a set of fuzzy rules such that the associated fuzzy controller approximates f to level ϵ?

The main result obtained by Castro is that the approximation is possible for almost any type of fuzzy logic controller.

4. Mamdani FLC with different inputs and implications

Further, the standard Mamdani FLC system will be extended to work as inputs with crisp data, intervals and linguistic terms and with various implications to represent the rules. A rule is characterized by

- a set of linguistic variables A, having as domain an interval $I_A = [a_A, b_A]$

- n_A linguistic values $A_1, A_2, ..., A_{n_A}$ for each linguistic variable A

- the membership function for each value A_i, denoted as $\mu^0_{A_i}(x)$ where $i \in \{1, 2, ..., n_A\}$ and $x \in I_A$.

The fuzzy inference process is performed in the steps presented in the following subsections.

4.1 Fuzzification

A fuzzification operator transforms a crisp data or an interval into a fuzzy set. For instance, $x_0 \in U$ is fuzzified into $\bar{x}_0$ according with the relation:

$$\mu_{\bar{x}_0}(x) = \begin{cases} 1 & if \quad x = x_0 \\ 0 & otherwise \end{cases}$$

and an interval input $[a, b]$ is fuzzified into

$$\mu_{[a,b]}(x) = \begin{cases} 1 & if \quad x \in [a, b] \\ 0 & otherwise \end{cases}$$

4.2 Firing levels

The firing level of a linguistic variable A_i, which appears in the premise of a rule, depends of the input data.

- For a crisp value x_0 it is $\mu^0_{A_i}(x_0)$.

If the input is an interval or a linguistic term then the firing level can be computed in various forms.

A) based on "intersection"

- for an input interval $[a, b]$ it is given by:

$$\mu_{A_i} = \max\{\min\{\mu^0_{A_i}(x), \mu_{[a,b]}(x)\} | x \in [a, b]\}.$$

- for a linguistic input value A'_i it is

$$\mu_{A_i} = \max\{\min\{\mu^0_{A_i}(x), \mu_{A'_i}(x)\} | x \in I_A\}.$$

B) based on "areas ratio"

- for an input interval $[a, b]$ it is given by the area defined by intersection $\mu^0_{A_i} \cap \mu_{[a,b]}$ divided by area defined by $\mu^0_{A_i}$

$$\mu_{A_i} = \frac{\int_a^b \min\{\mu^0_{A_i}(x), \mu_{[a,b]}(x)\}dx}{\int_a^b \mu^0_{A_i}(x)dx}$$

- for a linguistic input value A'_i it is computed as in the previous case

$$\mu_{A_i} = \frac{\int_a^b \min\{\mu^0_{A_i}(x), \mu_{A'_i}(x)\}dx}{\int_a^b \mu^0_{A_i}(x)dx}$$

It is obvious that, any t-norm T can be used instead of *min* and its dual t-conorm S instead of *max* in the previous formulas.

4.3 Fuzzy inference

The fuzzy control rules are of the form

$$R_i : IF\ X_1\ is\ A_i^1\ AND\ ...\ AND\ X_r\ is\ A_i^r\ THEN\ Y\ is\ C_i$$

where the variables $X_j, j \in \{1, 2, ..., r\}$, and Y have the domains U_j and V, respectively. The firing levels of the rules, denoted by $\{\alpha_i\}$, are computed by

$$\alpha_i = T(\alpha_i^1, ..., \alpha_i^r)$$

where T is a t-norm and α_i^j is the firing level for $A_i^j, j \in \{1, 2, ..., r\}$. The causal link from $X_1, ..., X_r$ to Y is represented using an implication operator I. It results that the conclusion C'_i inferred from the rule R_i is

$$\mu_{C'_i}(v) = I(\alpha_i, \mu_{C_i}(v)), \forall v \in V.$$

The formula

$$\mu_{C'}(v) = I(\alpha, \mu_C(v))$$

gives the following results, depending on the implication I:

Willmott : $\mu_{C'}(v) = I_W(\alpha, \mu_C(v)) = \max\{1 - \alpha, \min(\alpha, \mu_C(v))\}$

Mamdani: $\mu_{C'}(v) = I_M(\alpha, \mu_C(v)) = \min\{\alpha, \mu_C(v)\}$

Rescher-Gaines: $\mu_{C'}(v) = I_{RG}(\alpha, \mu_C(v)) = \begin{cases} 1 & if\ \ \alpha \leq \mu_C(v) \\ 0 & otherwise \end{cases}$

Kleene-Dienes: $\mu_{C'}(v) = I_{KD}(\alpha, \mu_C(v)) = \max\{1 - \alpha, \mu_C(v)\}$

Brouwer-Gödel: $\mu_{C'}(v) = I_{BG}(\alpha, \mu_C(v)) = \begin{cases} 1 & if\ \ \alpha \leq \mu_C(v) \\ \mu_C(v) & otherwise \end{cases}$

Goguen: $\mu_{C'}(v) = I_G(\alpha, \mu_C(v)) = \begin{cases} 1 & if\ \ \alpha \leq \mu_C(v) \\ \frac{\mu_C(v)}{\alpha} & otherwise \end{cases}$

Lukasiewicz: $\mu_{C'}(v) = I_L(\alpha, \mu_C(v)) = \min\{1 - \alpha + \mu_C(v), 1\}$

Fodor: $\mu_{C'}(v) = I_F(\alpha, \mu_C(v)) = \begin{cases} 1 & if\ \ \alpha \leq \mu_C(v) \\ \max\{1 - \alpha, \mu_C(v)\} & otherwise \end{cases}$

Reichenbach: $\mu_{C'}(v) = I_R(\alpha, \mu_C(v)) = 1 - \alpha + \alpha\mu_C(v)$

(a) Willmott implication

(b) Willmott implication

(c) Mamdani implication

(d) Rescher-Gaines implication

(e) Kleene-Dienes implication

(f) Brouwer-Gödel implication

(g) Goguen implication

(h) Lukasiewicz implication

(i) Fodor implication
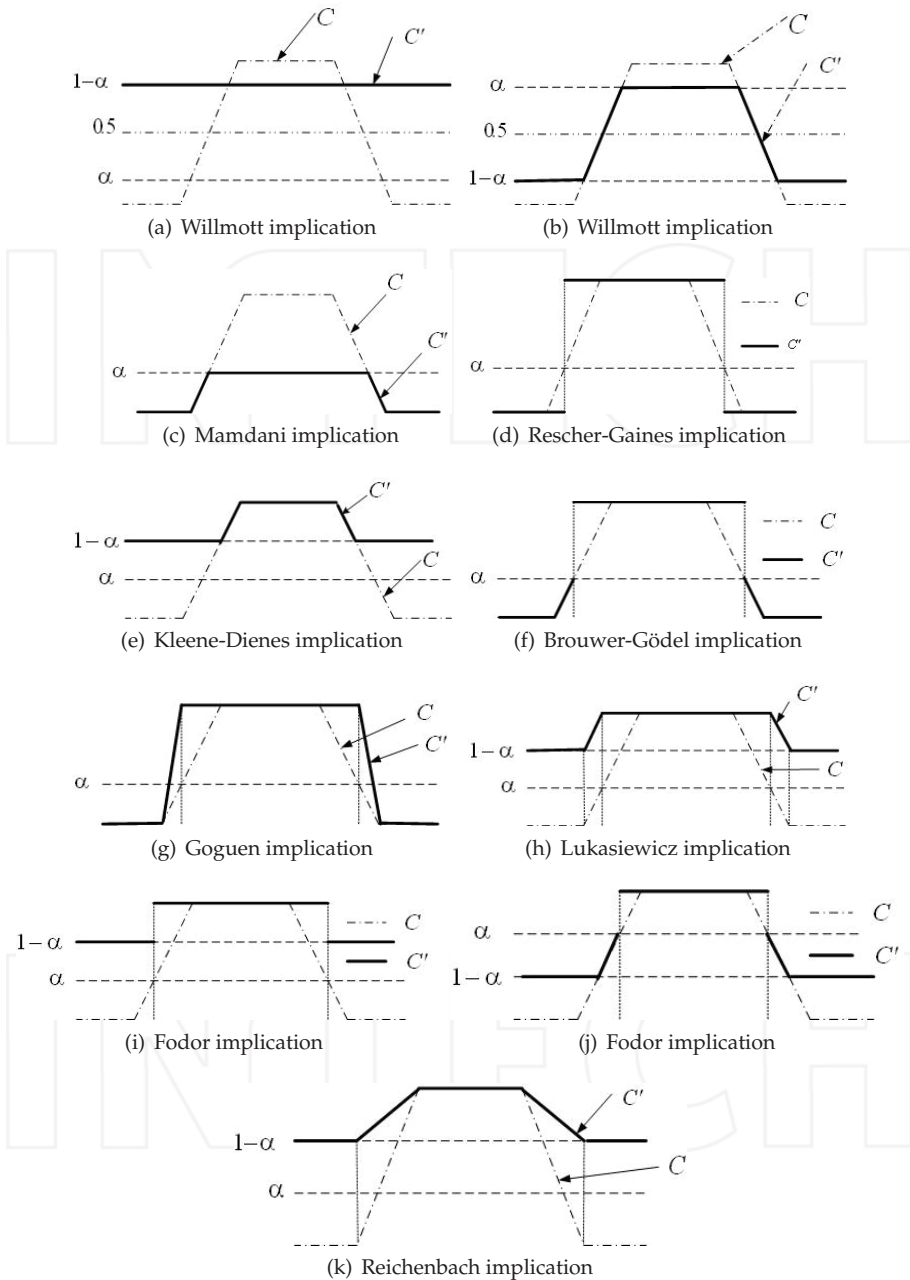
(j) Fodor implication

(k) Reichenbach implication

Fig. 5. Conclusions obtained with different implications

4.4 Defuzzification

The fuzzy output C'_i of the rule R_i is transformed into a crisp output z_i using the Middle-of-Maxima operator. The crisp value z_0 associated to a conclusion C' inferred from a rule having the firing level α and the conclusion C represented by the fuzzy number $(\underline{m}_C, \overline{m}_C, \alpha_C, \beta_C)$ is:

- $z_0 = \dfrac{\underline{m}_C + \overline{m}_C}{2}$ for implication $I \in \{I_R, I_{KD}\}$

- $z_0 = \dfrac{\underline{m}_C + \overline{m}_C + (1-\alpha)(\beta_C - \alpha_C)}{2}$ for $I \in \{I_M, I_{RG}, I_{BG}, I_G, I_L, I_F\}$ or $(I = I_W, \alpha \geq 0.5)$

- $z_0 = \dfrac{a_V + b_V}{2}$ if $I = I_W$, $\alpha < 0.5$ and $V = [a_V, b_V]$.

In the last case, in order to remain inside the support of C, one can choose a value according to Max-Criterion; for instance

$$z_0 = \frac{\underline{m}_C + \overline{m}_C + \alpha(\beta_C - \alpha_C)}{2}.$$

The overall crisp control action is computed by the discrete Center-of-Gravity method as follows. If the number of fired rules is N then the final control action is:

$$z_0 = \left(\sum_{i=1}^{N} \alpha_i z_i\right) / \sum_{i=1}^{N} \alpha_i$$

where α_i is the firing level and z_i is the crisp output of the i-th rule, $i = \overline{1, N}$.

Finally, the results obtained with various implication operators are combined in order to obtain the overall output of the system. For this reason, the "strength" $\lambda(I)$ of an implication I is used:

$$\lambda(I) = N(I)/13$$

where $N(I)$ is the number of properties (from the list $I1$ to $I13$) verified by the implication I (Iancu, 2009a). If the implications are considered in the order presented in the previous section, then according with the Definition 10, one obtains

$$w_1 = \lambda(I_W), w_2 = \lambda(I_M), ..., w_9 = \lambda(I_R)$$

$$a_1 = z_0(I_W), a_2 = z_0(I_M), ..., a_9 = z_0(I_R)$$

and the overall crisp action of the system is computed as

$$\overline{z}_0 = \sum_{j=1}^{9} w_j b_j$$

where b_j is the j-th largest element of $\{z_0(I_W), z_0(I_M), \ldots, z_0(I_F), z_0(I_R)\}$.

5. Applications

In order to show how the proposed system works, two examples will be presented. First example (Iancu, 2009b) is inspired from (Liu et al., 2005). A person is interested to buy a computer using on-line shopping. For this, the customer can make selections on the price and quality of computers. For the price of computers, different intervals are given for customers to choose from, for example, 0-200 EUR, 400-600 EUR, etc. For the quality of computers, five options are offered to the customers, namely Poor, Below Average, Average, Above Average, and Good. After customers make those selections, the satisfaction score for that selected computer is computed based on the fuzzy inference system described in the following. If customers are not satisfied with the satisfaction score, they can go back and make selections again. Therefore, this system will help customers to make decisions. In the next example the system works with two inputs and one output. The input variables are *quality* (Q) and *price* (P); the output variable is *satisfaction score* (S). The fuzzy rule base consist of

R_1: *IF Q is Poor AND P is Low THEN S is Middle*

R_2: *IF Q is Poor AND P is Middle THEN S is Low*

R_3: *IF Q is Poor AND P is High THEN S is Very Low*

R_4: *IF Q is Average AND P is Low THEN S is High*

R_5: *IF Q is Average AND P is Middle THEN S is Middle*

R_6: *IF Q is Average AND P is High THEN S is Low*

R_7: *IF Q is Good AND P is Low THEN S is Very High*

R_8: *IF Q is Good AND P is Middle THEN S is High*

R_9: *IF Q is Good AND P is High THEN S is Middle*

There are three linguistic values for the variable *price*:

$$\{Low, \ Middle, \ High\}$$

and five linguistic values for the variable *quality*:

$$\{Poor, \ Below \ Average, \ Average, \ Above \ Average, \ Good\}.$$

The universes of discourse are $[0, 800]$ for *price* and $[0, 10]$ for *quality*. The membership functions corresponding to the linguistic values are represented by the following trapezoidal fuzzy numbers:

$$Low = (0, 100, 0, 200)$$

$$Middle = (300, 500, 100, 100)$$

$$High = (700, 800, 200, 0)$$

$$Poor = (0, 1, 0, 2)$$

$$Below\ Average = (2, 3, 1, 1)$$

$$Average = (4, 6, 2, 2)$$

$$Above\ Average = (7, 8, 1, 1)$$

$$Good = (9, 10, 2, 0).$$

The *satisfaction score* has following linguistic values:

$$\{Very\ Low,\ Low,\ Middle,\ High,\ Very\ High\}$$

represented, in the universe $[0, 10]$, by the following membership functions:

$$Very\ Low = (0, 1, 0, 1)$$

$$Low = (2, 3, 1, 1.5)$$

$$Middle = (4, 6, 1, 1)$$

$$High = (7, 8, 1, 2)$$

$$Very\ High = (9, 10, 1, 0).$$

A person is interested to buy a computer with price = 400-600 EUR and quality = *AboveAverage*. The positive firing levels (based on intersection) corresponding to the linguistic values of the input variable *price* are

$$\mu_{Middle} = 1, \mu_{High} = 0.5$$

and the positive firing levels corresponding to the linguistic values of the input variable *quality* are:

$$\mu_{Average} = 2/3, \mu_{Good} = 2/3.$$

The fired rules and their firing levels, computed with t-norm Product, are:

$$R_5 \text{ with firing level } \alpha_5 = 2/3,$$

$$R_6 \text{ with firing level } \alpha_6 = 1/3,$$

$$R_8 \text{ with firing level } \alpha_8 = 2/3 \text{ and}$$

$$R_9 \text{ with firing level } \alpha_9 = 1/3.$$

Working with I_L implication, the fired rules give the following crisp values as output:
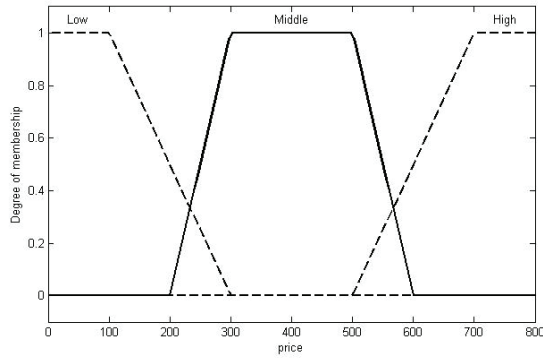
$$z_5 = 5, \ z_6 = 8/3, \ z_8 = 23/3, \ z_9 = 5;$$
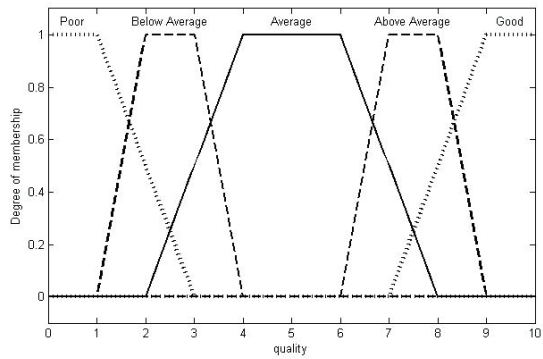
the overall crisp control action for I_L is

$$z_0(I_L) = 5.5$$

Working with I_R implication, the fired rules give the following crisp values as output:
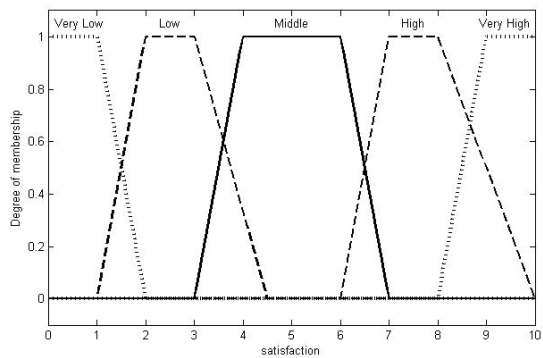
$$z_5 = 5, \ z_6 = 2.5, \ z_8 = 7.5, \ z_9 = 5;$$

(a) The membership function of the variable *price*

(b) The membership function of the variable *quality*

(c) The membership function of the variable *satisfaction score*

Fig. 6. Membership functions for the input and output variables

its overall crisp action is

$$z_0(I_R) = 5.416.$$

Because $\lambda(I_R) = 11/13$ and $\lambda(I_L) = 1$, the overall crisp action given by system is

$$\overline{z}_0 = 5.4615$$

The standard Mamdani model applied for this example gives the following results:

- the firing levels are: $\alpha_5 = 2/3$, $\alpha_6 = 0.5$, $\alpha_8 = 2/3$, $\alpha_9 = 0.5$
- the crisp rules outputs are: $z_5 = 5$, $z_6 = 5.25/2$, $z_8 = 23/3$, $z_9 = 5$
- the overall crisp action is: $z_0 = 23/3 = 7.66$

If the Center-of-Gravity method (instead of maximum operator) is used to compute the overall crisp action then $z_0 = 5.253$

One observes an important difference between these two results and also between these results and those given by our method. An explanation consists in the small value of the "strength" of Mamdani's implication in comparison with the values associated with Reichenbach and Lukasiewicz implications; the strength of an implication is a measure of its quality. From different implications, different results will be obtained if separately implications will be used. The proposed system offers a possibility to avoid this difficulty, by aggregation operation which achieves a "mediation" between the results given by various implications.

Another application that uses this type of controller is presented in (Iancu, 2009b) concerning washing machines (Agarwal, 2007). When one uses a washing machine, the person generally select the length of washing time based on the amount of clothes he/she wish to wash and the type and degree of dirt cloths have. To automate this process, one uses sensors to detect these parameters and the washing time is then determined from this data. Unfortunately, there is no easy way to formulate a precise mathematical relationship between volume of clothes and dirt and the length of washing time required. Because the input/output relationship is not clear, the design of a washing machine controller can be made using fuzzy logic. A fuzzy logic controller gives the correct washing time even though a precise model of the input/output relationship is not available.

The problem analyzed in this example has been simplified by using only two inputs and one output. The input variables are *degree-of-dirt (DD)* and *type-of-dirt (TD)*; the output variable is *washing-time (WT)*. The fuzzy rule-base consist of:

R_1: *IF DD is Large AND TD is Greasy THEN WT is VeryLong*

R_2: *IF DD is Medium AND TD is Greasy THEN WT is Long*

R_3: *IF DD is Small AND TD is Greasy THEN WT is Long*

R_4: *IF DD is Large AND TD is Medium THEN WT is Long*

R_5: *IF DD is Medium AND TD is Medium THEN WT is Medium*

R_6: *IF DD is Small AND TD is Medium THEN WT is Medium*

R_7: *IF DD is Large AND TD is NotGreasy THEN WT is Medium*

R_8: *IF DD is Medium AND TD is NotGreasy THEN WT is Short*

R_9: *IF DD is Small AND TD is NotGreasy THEN WT is VeryShort*

There are three linguistic values for the variable *degree-of-dirt*:

$$\{Small,\ Medium,\ Large\}$$

and five linguistic values for the variable *type-of-dirt*:

$$\{VeryNotGreasy,\ NotGreasy,\ Medium,\ Greasy,\ VeryGreasy\}$$

having the same universe of discourse: $[0, 100]$. The membership functions corresponding to the linguistic values are represented by the following trapezoidal fuzzy numbers:

$$Small = (0, 20, 0, 20)$$

$$Medium = (40, 60, 20, 20)$$

$$Large = (80, 100, 20, 0)$$

$$VeryNotGreasy = (0, 10, 0, 20)$$

$$NotGreasy = (20, 30, 10, 10)$$

$$Medium = (40, 60, 20, 20)$$

$$Greasy = (70, 80, 10, 10)$$

$$VeryGreasy = (90, 100, 20, 0).$$

The *washing-time* has following linguistic values

$$\{VeryShort,\ Short,\ Medium,\ Long,\ VeryLong,\ High\}$$

represented in the universe $[0, 60]$ by the membership functions:

$$VeryShort = (0, 5, 0, 5)$$

$$Short = (10, 15, 10, 5)$$

$$Medium = (20, 30, 5, 5)$$

$$Long = (35, 50, 5, 10)$$

$$VeryLong = (50, 60, 10, 0).$$

A person is interested to wash some clothes with the *degree-of-dirt* between 60 and 70 and *type-of-dirt* is *VeryGreasy*. Working in the same conditions as in the previous example, but using "areas ratio" instead of "intersection" in order to compute the firing levels, one obtains the following results. The positive firing levels corresponding to the linguistic values of the input variable *degree-of-dirt* are
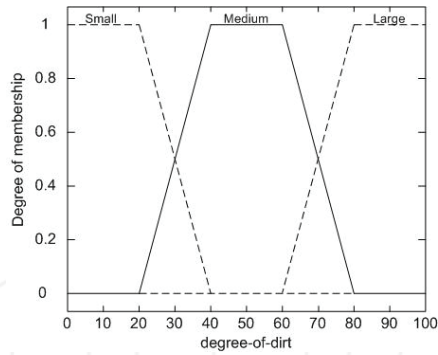
(a) The membership function of the variable *degree-of-dirt*
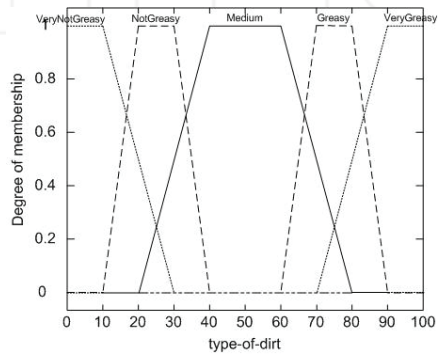
(b) The membership function of the variable *type-of-dirt*
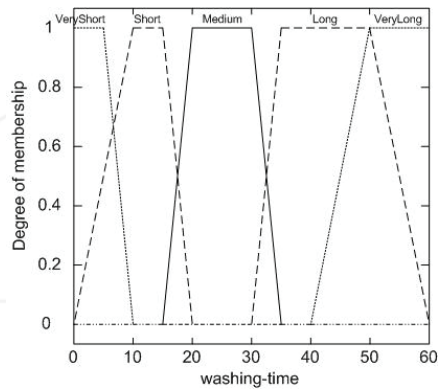
(c) The membership function of the variable *washing-time*

Fig. 7. Membership functions for the input and output variables

$$\mu_{Medium} = 0.1875, \mu_{Large} = 0.0833$$

and the positive firing levels corresponding to the linguistic values of the input variable *type-of-dirt* are:

$$\mu_{Medium} = 0.0312, \mu_{Greasy} = 1/3.$$

The fired rules and their firing levels, computed with t-norm Product, are:

$$R_1 \text{ with firing level } \alpha_1 = 0.0277,$$

$$R_2 \text{ with firing level } \alpha_2 = 0.0625,$$

$$R_4 \text{ with firing level } \alpha_4 = 0.0026 \text{ and}$$

$$R_5 \text{ with firing level } \alpha_5 = 0.0058.$$

Working with I_L implication, the fired rules give the following crisp values as output:

$$z_1 = 50.138, \; z_2 = 44.843, \; z_4 = 44.993, \; z_5 = 25$$

and the overall crisp control action for I_L is

$$z_0(I_L) = 45.152$$

Working with I_R implication, the fired rules give the following crisp values as output:

$$z_1 = 55, \; z_2 = 42.5, \; z_4 = 42.5, \; z_5 = 25$$

and its overall crisp control action is

$$z_0(I_R) = 44.975$$

The overall crisp action given by system is computed using the technique OWA with

$$w_1 = 13/24, w_2 = 11/24, a_1 = 45.152, a_2 = 44.975;$$

it results

$$\bar{z}_0 = 45.079$$

6. Conclusion

This paper presents fuzzy logic controllers of Mamdani type. After the standard Mamdani FLC is explained, an its extension is prezented. Because it can work not only with crisp data as inputs but, also, with intervals and/or linguistic terms its area of applications is very large. As it is mentioned in (Liu et al., 2005), a very important domain of its application is WEB shopping. Web users may use convenient interval inputs for online shopping as in the previous example. The working with various implications in the same time and, moreover, the possibility to aggregate the results given by these implications offer a strong base for more accurate results of our system.

The system can be improved by adding new implications, by using other fuzzy matching techniques or by other aggregate operators in order to obtain a overall crisp action from those given, separately, by every implication.

One of our future preoccupation is to extend this system by incorporating uncertainty about the membership functions of fuzzy sets associated with linguistic terms. For this we intend to work with interval type-2 fuzzy sets in accordance with the results from Mendel (2001; 2003; 2007).

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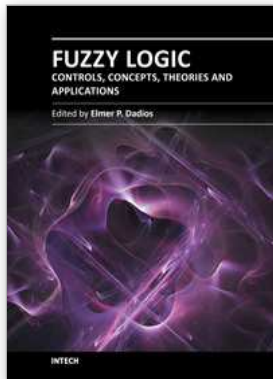
Fuzzy Logic - Controls, Concepts, Theories and Applications
Edited by Prof. Elmer Dadios

This book introduces new concepts and theories of Fuzzy Logic Control for the application and development of robotics and intelligent machines. The book consists of nineteen chapters categorized into 1) Robotics and Electrical Machines 2) Intelligent Control Systems with various applications, and 3) New Fuzzy Logic Concepts and Theories. The intended readers of this book are engineers, researchers, and graduate students interested in fuzzy logic control systems.

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